MORSE AND GENERIC CONTACT BETWEEN FOLIATIONS

BY

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ABSTRACT. Motivated by the recent work of J. Franks and C. Robinson, the study of the contact between two foliations of equal codimension is begun. Two foliations generically contact each other in certain dimensional submanifold complexes. All but a finite number of these contact points are "Morse". In a recent paper by the author, a complete large isotopy "index of contact" is specified for two foliations of T^2. If contact is restricted to index 0 ("domed contact"), some sharp conclusions are made as to the topology of the manifold and isotopy classes of the two foliations. It is hoped that this work will lead to the construction of new quasi-Anosov diffeomorphisms and possibly to a new Anosov diffeomorphism.

0. Preliminaries.

Introduction. In §1, contact points (i.e., the points where the two foliations are not transverse) are shown to be either nice (Morse) or not so nice (non-Morse). A generalized "Lemma of Morse" is proved. Then the generic situation (isotopy topology) is worked out in Theorem 1. For example, two codimension 1 foliations of a compact manifold generically have contact, the finite union of circles; furthermore, all but a finite number of these contact points are Morse. The proof is complicated by the fact that the plane fields must be perturbed through integrable plane fields.

To produce a quasi-Anosov diffeomorphism which was not Anosov, John Franks and Clark Robinson [1] needed to isotope two 1-dimensional foliations of $S^2 \times I$, nontangent (or "clean"). This isotopy had to be carefully and painstakingly constructed. An invariant of "isotransversability" is not available except on T^2 where the complete large isotopy problem is known [14]. But in higher dimensions, the situation is still overbearing.

The last section contains a sharp result about two foliations of a compact manifold with boundary where the contact is restricted to "domed" contact. Such contact can only occur between foliations of thick spheres one of which is the trivial foliation, the other an element of one of five certain isotopy classes.

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DEFINITIONS. A good reference for background material and basic definitions is Blaine Lawson's survey article [5].

A C^∞ smooth *codimension k foliation* $\mathcal{F}$ of a manifold M^n is a collection of *leaves* $\{\mathcal{L}\}$ such that for all $x \in M$ there exists a locally defined smooth *foliation chart* $f\colon U \to \mathbf{R}^k$, $x \in U$, for which all $y \in U$ are regular points and such that $\{\mathcal{L} \cap U\colon \mathcal{L} \in \mathcal{F}\}$ are the level surfaces of f. If M is a manifold with boundary then $\mathcal{F}$ is a *proper foliation* of M if $\mathcal{F}$ foliates int M and $\mathcal{F} \cap \partial M$ is a codimension $(k-1)$ foliation. So if $k = 1$, the boundary components are leaves of $\mathcal{F}$.

$\mathcal{F}$ is a *foliation with boundary* of M if $\mathcal{F}$ may be extended to a foliation of an open neighborhood of M (M thought of as being embedded in some Euclidean space).

A codimension 1 foliation $\mathcal{F}$ is *transversely orientable* if there exists a nonsingular vector field transverse to $\mathcal{F}$. In this paper all codimension 1 foliations will be transversely orientable. A *trivial* foliation of $M \times N$ has leaves of the form $M \times \{x\}$, $x \in N$. U is a *foliation neighborhood* of a leaf $\mathcal{L} = M \times \{x_0\}$ of a trivial foliation if $U = M \times B(x_0)$ (an open ball about x_0 contained in N) and U is foliated by $\{M \times \{x\}\colon x \in B(x_0)\}$.

Associated to a codimension k foliation $\mathcal{F}$ is a *tangent k-plane field* $\sigma_{\mathcal{F}}$. These plane fields are sections of the $(n, n-k)$-Grassmannian bundle over M:

$$G_{n,n-k} \quad \hookrightarrow \quad GB_{n,n-k}$$
$$\sigma_{\mathcal{F}} \uparrow \downarrow \pi$$
$$M$$

Plane fields associated to foliations must satisfy the Frobenius conditions; such plane fields are called *integrable*. Two foliations $\mathcal{F}$ and $\mathcal{G}$ of M are C^∞-*close* if $\sigma_{\mathcal{F}}$ and $\sigma_{\mathcal{G}}$ are C^∞-close. Designate by $\mathrm{Fol}_k(M)$ the set of all C^∞ smooth codimension k foliations with the C^∞-*topology*. P is a *generic* property of foliations $\mathcal{F} \in \mathrm{Fol}_k(M)$ if P holds for a residual subset of $\mathrm{Fol}_k(M)$.

Two foliations $\mathcal{F}$ and $\mathcal{G}$ are *isotopic* or $\mathcal{F} \sim \mathcal{G}$ if there exists a diffeomorphism f isotopic to the identity such that $f(\mathcal{F}) = \mathcal{G}$. In Lawson [6], the term *completely equivalent* has the same meaning. Two foliations are *homotopic* if their plane fields are homotopic through integrable plane fields. The homotopy class of $\mathcal{F}$ is designated by $[\mathcal{F}]$.

For $\mathcal{F}$ and $\mathcal{G}$ two codimension k foliations of M^n, let $T_x\mathcal{F}$ denote the tangent space to the leaf of $\mathcal{F}$ through x, similarly for $T_x\mathcal{G}$ (so $T_x\mathcal{F}$ and $T_x\mathcal{G}$ are both $(n-k)$-dimensional). For $2k \leqslant n$, $\mathcal{F}$ and $\mathcal{G}$ are *transverse*, if $T_x\mathcal{F} + T_x\mathcal{G} = T_xM$ (at each $x \in M$). For unrestricted k, $\mathcal{F}$ and $\mathcal{G}$ are *clean*

if $\dim(T_x \mathcal{F} + T_x \mathcal{G}) = \min(n, 2(n - k))$, for each $x \in M$ (so $\mathcal{F}$ and $\mathcal{G}$ are as "transverse" as possible). Transverse implies clean. The set of points of M where $\mathcal{F}$ and $\mathcal{G}$ are not clean is called the *contact set* or $\mathrm{Con}(\mathcal{F}, \mathcal{G})$. The set of points where $\mathcal{F}$ and $\mathcal{G}$ are tangent is called the *tangency set* or $\mathrm{Tan}(\mathcal{F}, \mathcal{G})$. $\mathcal{F}$ and $\mathcal{G}$ are *isocleanable* if there exists an isotopy $h_t \colon M \hookleftarrow$ such that $h_1(\mathcal{F})$ is clean to $\mathcal{G}$. Similarly for *isotransversable*. A *Reeb sector* is a solid cylinder $D^{n-1} \times I$ which has a foliation with boundary $\mathcal{F}$ defined by $f(x_1, \ldots, x_n) = (x_1^2 + x_2^2 + \cdots + x_{n-1}^2) - x_n$ and is such that $\partial D^{n-1} \times I$ is a leaf of $\mathcal{F}$. (See Figure 1.)
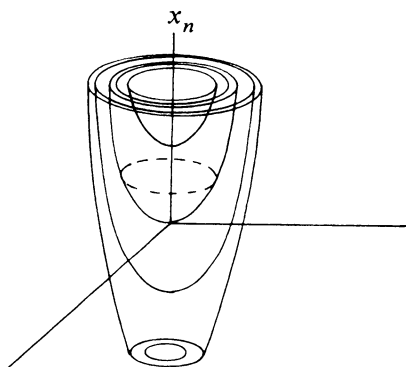

FIGURE 1

Two codimension 1 foliations of M, $\mathcal{F}$ and $\mathcal{G}$, are *integrably homotopic* if there exists a foliation with boundary, $\tilde{\mathcal{F}}$, of $M \times I$ such that (i) $\tilde{\mathcal{F}} \cap \{M \times \{0\}\} = \mathcal{F}$ and $\tilde{\mathcal{F}} \cap \{M \times \{1\}\} = \mathcal{G}$, and (ii) $\tilde{\mathcal{F}}$ is transverse to the trivial foliation $\{M \times \{t\} \colon t \in I\}$. A theorem of Milnor [8] states that integrably homotopic is equivalent to isotopic. If the transversality condition is dropped then $\mathcal{F}$ and $\mathcal{G}$ are merely *concordant*. Any two foliations of T^2 are concordant (Thurston [13]).

The symbol "$\approx$" denotes diffeomorphic (e.g., $M \approx S^{n-1} \times I$). $\chi(M)$ is the Euler characteristic of M. All manifolds are smooth and orientable. The *double manifold* of a manifold M with boundary equals M glued to a copy of M along ∂M via $\mathrm{id}_{\partial M}$.

Reeb stability: The following Reeb stability theorem ([10], [6]) is the most important theorem on compact leaves and their topological type.

THEOREM (G. REEB). *Let $\mathcal{F}$ be a C^r foliation on a manifold M, $r \geqslant 0$, and suppose that $\mathcal{F}$ has a compact leaf $\mathcal{L}$ with finite fundamental group. Then every neighborhood of $\mathcal{L}$ contains an invariant neighborhood U with the property that each leaf $\mathcal{L}' \subset U$ is a finite covering space of $\mathcal{L}$.*

Thus, if $\mathcal{L}$ is simply connected, U has the form $\mathcal{L} \times D^q$ with the foliation $\{\mathcal{L} \times \{x\}: x \in D\}$. M need not be compact; the following is a strengthening in the codimension 1, M compact case:

COROLLARY [10], [6]. *Let M be a compact manifold with a smooth codimension 1, transversely orientable foliation $\mathcal{F}$. If $\mathcal{F}$ has a compact leaf $\mathcal{L}$ with finite fundamental group, then M is a fiber bundle over S^1 with fiber $\mathcal{L}$.*

1. Morse contact. It will be shown in Theorem 1(iv) that generically points of contact between two codimension 1 foliations are points of Morse contact.

DEFINITION. Assume that $\mathcal{F}$ and $\mathcal{G}$ are codimension k foliations of M and that $f: U \to \mathbf{R}^k$ defines $\mathcal{F}$ locally near x_0. Assume $x_0 \in \mathcal{L} \in \mathcal{G}$. Then $\mathcal{F}$ has *Morse contact* with $\mathcal{G}$ *at* x_0 if $G = f|_{\mathcal{L}}$ has a critical point at x_0 and the following $n \times kn$ matrix of second partial derivatives has maximal rank:

$$\left(\frac{\partial^2 G_1}{\partial x_i \partial x_j} \quad \frac{\partial^2 G_2}{\partial x_i \partial x_j} \quad \cdots \quad \frac{\partial^2 G_k}{\partial x_i \partial x_j} \right).$$

Here the G_i are the coordinate functions of G and the $\{x_i\}$ are local coordinates on U. If all points of $\mathrm{Con}(\mathcal{F}, \mathcal{G})$ are Morse points then $\mathcal{F}$ has *Morse contact* with $\mathcal{G}$.

If $k = 1$, then G has a Morse critical point at x_0. If this critical point is index p or index $(n - p)$, then $\mathcal{F}$ has *index p Morse contact* with $\mathcal{G}$ at x_0: Morse contact is intrinsic and does not depend on the choice of f.

EXAMPLES. (1) $\mathcal{F} = $ 0-sloped foliation of T^2, $\mathcal{G}$ is a snaking isotope of $\mathcal{F}$ (Figure 2). All contact is index 0.
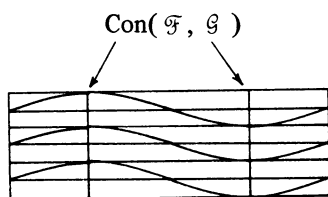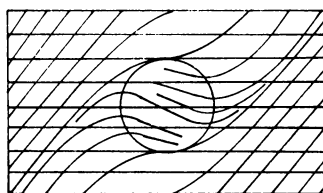

FIGURE 2   FIGURE 3

(2) $\mathcal{F} = $ the 0-sloped foliation of T^2, $\mathcal{G}$ is an isotope of the 1-sloped foliation. $\mathrm{Con}(\mathcal{F}, \mathcal{G})$ is a circle. (Figure 3.) Two contact points are not Morse. Such points will be called *non-Morse* points. There is no foliation near $\mathcal{G}$ which has Morse contact with $\mathcal{F}$.

(3) $\mathcal{F}$ foliates $\mathbf{R}^3$ and is defined $f(x, y, z) = z$. $\mathcal{G}$ is a 1-parameter family of parabolas, defined by $g(x, y, z) = x^2 + y^2 - z^2$. (Figure 4.) Points on the z-axis are index 0 Morse contact points.
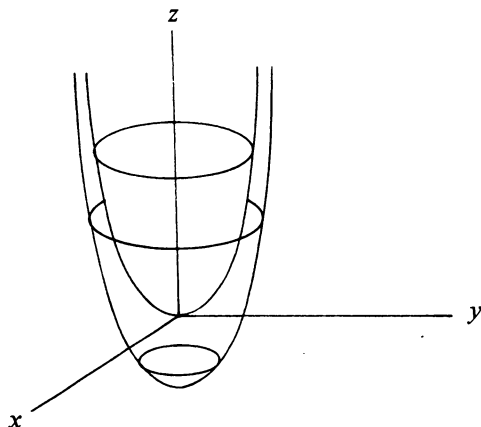
FIGURE 4

Codimension 1. When both foliations are codimension 1, a generalization of the "Lemma of Morse" (Milnor [9]) can be proved.

MORSE CONTACT LEMMA. *Assume two codimension* 1 *foliations* $\mathcal{F}$ *and* $\mathcal{G}$ *are locally defined on* U *by* f *and* g. *Assume* $\mathcal{F}$ *and* $\mathcal{G}$ *have Morse contact at* $p \in U$. *Then on a smaller neighborhood* $V \subset U$ *of* p, *there exist smooth coordinates* $(x_1, \ldots, x_n)$ *such that* $p = (0, \ldots, 0)$, $f = x_n$, *and*

$$g = x_n - \left(x_1^2 + x_2^2 + \cdots + x_\lambda^2\right) + \left(x_{\lambda-1}^2 + \cdots + x_{n-1}^2\right).$$

PROOF. First we show that $\mathrm{Con}(\mathcal{F}, \mathcal{G}) \pitchfork \mathcal{F}$ at x_0. We may assume that there exist smooth coordinates $(x_1, \ldots, x_n)$ on U such that $f = x_n$ and $p = (0, \ldots, 0)$.

$\mathcal{F}$ and $\mathcal{G}$ being codimension 1, their plane fields $\sigma_1 = \sigma_{\mathcal{F}}$ and $\sigma_2 = \sigma_{\mathcal{G}}$ may be considered as maps into $\mathbf{P}^{n-1}$. In homogeneous coordinates

$$\sigma_1 = \left[\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right] = [0, 0, \ldots, 0, 1] \quad \text{and} \quad \sigma_2 = \left[\frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n}\right].$$

Since σ_2 is nonsingular we may assume that $\partial g/\partial x_n$ is nonzero. This allows us to change to inhomogeneous coordinates by dividing through by the nth-coordinate, then projecting onto the first $(n-1)$-coordinates. Then

$$\sigma_1 = [0, \ldots, 0] \quad \text{and} \quad \sigma_2 = \left[\frac{\partial g/\partial x_1}{\partial g/\partial x_n}, \ldots, \frac{\partial g/\partial x_{n-1}}{\partial g/\partial x_n}\right]$$

and are considered as maps into $\mathbf{R}^{n-1}$.

Restrict σ_1 and σ_2 to $U \cap \mathcal{L}$ where $p \in \mathcal{L} \in \mathcal{F}$. Now assume there exists nonzero $v_0 = \sum_{j=1}^{n-1} a_j \partial/\partial x_j \in T_p\mathrm{Con} \cap T_p\mathcal{L}$. Two codimension 1 foliations have contact precisely where they are tangent. So $\mathrm{Con}(\mathcal{F}, \mathcal{G}) = \{x \in M: (\sigma_1 - \sigma_2)(x) = 0\}$. If $v_0 \in T_p\mathrm{Con}$ then $v_0(\sigma_1 - \sigma_2) = 0$. But in inhomogeneous coordinates

$$v_0(\sigma_1 - \sigma_2)(p) = \left(\sum a_j \frac{\partial}{\partial x_j} \left(-\frac{\partial g/\partial x_1}{\partial g/\partial x_n} \right), \ldots, \sum a_j \frac{\partial}{\partial x_j} \left(-\frac{\partial g/\partial x_{n-1}}{\partial g/\partial x_n} \right) \right)_p$$

$$= \frac{1}{\partial g/\partial x_n} \left(\begin{array}{c} \text{Hessian of } g \\ \text{at } p \end{array} \right) \cdot v_0 \neq 0 \ \text{ iff } g \text{ is Morse on } U \cap \mathcal{L}.$$

So no such v_0 exists.

It follows from Theorem 1(i) (independently) that $\mathrm{Con}(\mathcal{F}, \mathcal{G})$ is a finite union of 1-dimensional submanifolds. From the above argument, $\mathrm{Con}(\mathcal{F}, \mathcal{G})$ is transverse to $\mathcal{L}$ at p. So there exist coordinates again called $(x_1, \ldots, x_n)$ such that $f = x_n$ and such that $\mathrm{Con}(\mathcal{F}, \mathcal{G}) \cap U$ occupies the x_n-axis. Since $g|_{\mathcal{L}}$ is Morse at p, there exists a foliation neighborhood N about $\mathcal{L}$ such that for all $\mathcal{L} \in N$, $g|_{\mathcal{L}}$ is Morse at $\mathcal{L} \cap \mathrm{Con}(\mathcal{F}, \mathcal{G})$. The remainder of the proof mimics Milnor's proof of the Lemma of Morse considering instead of a single Morse function f, a one-parameter family of Morse functions f_μ each with Morse critical points on the x_n-axis.   Q.E.D.

Codimension k. Assume two codimension k foliations $\mathcal{F}$ and $\mathcal{G}$ are defined locally by $f, g: U \to \mathbf{R}^k$ where $f = (f_1, \ldots, f_k)$ and $g = (g_1, \ldots, g_k)$.

Conjecture (*Codimension k Morse Contact Lemma*). If p is a point of Morse contact between $\mathcal{F}$ and $\mathcal{G}$ then there exist coordinates $(x_1, \ldots, x_n)$ near p such that $f_1 = x_n, f_2 = x_{n-1}, \ldots, f_k = x_{n-k-1}$ and

$$g_j = x_{n-j+1} - G_j(x_1, \ldots, x_{n-j-1}) \quad \text{for } 1 \leqslant j \leqslant k.$$

As indicated by Smale [11], one cannot expect each of these G_j to be Morse functions. If p is also required to be a Pareto optimal point for the set of functions $\{ g_1|_{\mathcal{L}}, \ldots, g_k|_{\mathcal{L}} \}$ $(p \in \mathcal{L} \in \mathcal{F})$, then perhaps these G_j may be Morse.

2. Generic contact. Isotransversability is not a dense property, nor is it always possible to perturb a foliation so that it has Morse contact with a fixed foliation. Generally, two foliations have as contact set the finite union of submanifold complexes (see Hirsch [3]).

DEFINITION. Submanifolds $N_0, \ldots, N_q \subset M$ form a *submanifold complex* of dimension q if (i) N_0 is closed and $\overline{N}_{i+1} \setminus N_{i+1} \subset N_0 \cup \cdots \cup N_i$; (ii) $N_q \neq \varnothing$; (iii) $\dim N_{i-1} < \dim N_i$; (iv) let $0 \leqslant i, j \leqslant q$, $d = \dim N_i$; then if a sequence $\{x_n\}$ in N_j converges to y in N_i, there is a sequence of d-planes $E_n \subset T_{x_n} N_j$ converging to $T_y N_i$.

This last condition assures that in the generic case a submanifold and a submanifold complex intersect transversally (i.e., the submanifold is transverse to each submanifold of the complex). An example of a 2-dimensional submanifold complex of $\mathbf{R}^3$ is the "sphere with lips". (Figure 5.)
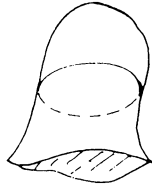
FIGURE 5

We now define two special submanifold complexes, $\mathcal{C}_\sigma$ and $\mathcal{B}_\sigma$, of the Grassmannian bundle associated to an $(n - k)$-plane field σ.

$$G_{n,n-k} \hookrightarrow \quad GB_{n,n-k} \quad \supset \mathcal{C}_\sigma, \mathcal{B}_\sigma$$
$$\sigma \uparrow \downarrow$$
$$M$$

For $k \leqslant n/2$, $\mathcal{C}_\sigma = \bigcup_{x \in M} A_x$ where

$$A_x = \{ W \in G_{n,n-k}(x) : W \text{ is not transverse to } \sigma(x) \}.$$

For $k \geqslant n/2$, $\mathcal{B}_\sigma = \bigcup_{x \in M} B_x$ where

$$B_x = \{ W \in G_{n,n-k}(x) : \dim(W \oplus \sigma(x)) < 2(n - k) \}.$$

These are submanifold complexes because, for each x, A_x and B_x are naturally diffeomorphic to the following spaces of linear maps:

$$A_x \approx \{ \phi \in L(V^{n-k}, V^k) : \phi \text{ has rank } \leqslant k - 1 \},$$

$$B_x \approx \{ \phi \in L(V^{n-k}, V^k) : \phi \text{ is not injective} \},$$

where V^{n-k} is an $(n - k)$-dimensional subspace of an n-dimensional vector space and where by V^k we mean $V^{n-k\perp}$. Each A_x is a union of submanifolds, $A_x^s \approx \{ \phi : \text{rank } \phi = s \}$, $0 \leqslant s \leqslant k - 1$. Similarly for each B_x. It follows then from Hirsch [3] that A_x is a submanifold complex of dimension $(k - 1)(n - k + 1)$. Thus $\mathcal{C}_\sigma$ is a $[(k - 1)(n - k + 1) + n]$-submanifold complex. Similarly $\mathcal{B}_\sigma$ is $[(n - k - 1)(k + 1) + n]$-dimensional.

THEOREM 1 (GENERIC CONTACT). *Fix* $\mathcal{F}_0 \in \text{Fol}_k^\infty(M)$, *M a smooth compact n-manifold* $(\sigma_0 = \sigma_{\mathcal{F}_0})$.

(i) *For* $k \leqslant n/2$ *let* $\mathcal{R} = \mathcal{R}(\mathcal{F}_0) = \{ \mathcal{F} \in \text{Fol}_k(M) : \sigma_{\mathcal{F}} \pitchfork \mathcal{C}_{\sigma_0} \}$. *Then*

(a) $\mathcal{R}$ *is open and dense in* $\text{Fol}_k^\infty(M)$, *and*

(b) *for all* $\mathcal{F} \in \mathcal{R}$, $\text{Con}(\mathcal{F}, \mathcal{F}_0)$ *is the finite union of smooth* $(2k - 1)$-*dimensional submanifold complexes.*

(ii) *For* $k \geqslant n/2$, *let* $\mathcal{R} = \mathcal{R}(\mathcal{F}_0) = \{ \mathcal{F} \in \text{Fol}_k^\infty(M) : \sigma_{\mathcal{F}} \pitchfork \mathcal{B}_{\sigma_0} \}$. *Then*

(a) $\mathcal{R}$ *is open and dense in* $\text{Fol}_k^\infty(M)$, *and*

(b) $\text{Con}(\mathcal{F}, \mathcal{F}_0)$ *is the finite union of smooth* $[2(n - k) - 1]$-*dimensional submanifold complexes.*

(iii) *For all $\mathcal{F}$ and $\varepsilon > 0$, there exists a smooth isotopy h_t: $M \hookrightarrow$, C^∞ ε-near to the identity such that $h_1(\mathcal{F}) \in \mathcal{R}$.*

(iv) *When $k = 1$ there exists an open and dense subset $\mathcal{R}_0$ of $\mathcal{R}(\mathcal{F}_0)$ such that for all $\mathcal{F} \in \mathcal{R}_0$, all but a finite number of points of $\mathrm{Con}(\mathcal{F}, \mathcal{F}_0)$ are Morse. Furthermore, there exists a C^∞-small isotopy which carries any arbitrary foliation to an element of $\mathcal{R}_0$.*

DEFINITION. Two codimension k foliations $\mathcal{F}$ and $\mathcal{G}$ have *generic contact* ($k > 1$) if $\mathcal{F} \in \mathcal{R}(\mathcal{G})$. If $k = 1$, we require that $\mathcal{F} \in \mathcal{R}_0(\mathcal{G})$.

The proof of this theorem is complicated by the fact that perturbations of $\sigma_{\mathcal{F}}$ in $G\mathcal{B}_{n,n-k}$ must pass through integrable plane fields. The key lemma which assures that this sort of perturbation is possible is due to Lawson [7]:

LEMMA 2. *Let U, M and $\tilde{M}$ be smooth manifolds and $S \subset \tilde{M}$ a smooth submanifold; and let F: $U \times M \to \tilde{M} \supset S$ be a smooth map. If F is transverse to S, then the mapping F_u: $M \to \tilde{M}$ given by $F_u(x) = F(u, x)$ is also transverse to S for a.a. $u \in U$.*

This lemma is applied in our case as follows:

LEMMA 2'. *Let N_1 be a submanifold and N_2 be a submanifold complex of $G_{n,n-k} \times \mathbf{R}^n$. Then for a.a. $g = (g_1, g_2) \in \mathcal{O}_n \times E^n$ (the orthogonal and Euclidean groups), $g(N_1) \pitchfork N_2$.*

PROOF. First, Lemma 2 may be modified to allow S to be a submanifold complex; by Hirsch [3], it is an open and dense property that a submanifold and a submanifold complex meet transversally (or cleanly).

Next, let $\tilde{M} = G_{n,n-k} \times \mathbf{R}^n$. Define μ: $(\mathcal{O}_n \times E^n) \times N_1 \to \tilde{M} \supset N_2$ by μ: $(g, x) \mapsto g(x)$. The $(\mathcal{O}_n \times E^n)$-action on $G_{n,n-k} \times \mathbf{R}^n$ is transitive; thus μ is a submersion. Trivially, $\mu \pitchfork N_2$. Then, for a.a. g, μ_g: $N_1 \to M$ is transverse to N_2 (or clean to N_2) where $\mu_g = \mu(g, x) = g(x)$.   Q.E.D.

PROOF OF THEOREM 1. That $\mathcal{R}$ is open for all k is a consequence of transversality theory; we now show density. Let $\{V_i \subset U_i\}$ be a finite collection of open balls of radii $r/2$ and r, respectively, which cover M. Because each V_i is diffeomorphic to $\mathbf{R}^n$, the action $g \in \mathcal{O}_n \times E^n$ of Lemma 2' naturally induces an action on the Grassmannian bundle over V_i:

$$G_{n,n-k} \hookrightarrow \quad GB_{n,n-k} \quad \approx \quad G_{n,n-k} \times V_i$$
$$\uparrow\downarrow \sigma|_{V_i}, \sigma_0|_{V_i}$$
$$V_i$$

Let $N_1 = \sigma(V_i) = \sigma_{\mathcal{F}}(V_i)$ and let $N_2 = \mathcal{C}_{\sigma_0}$ when $k \leqslant n/2$ or let $N_2 = \mathcal{B}_{\sigma_0}$ when $k \geqslant n/2$. By Lemma 2', for a.a. $g \in \mathcal{O}_n \times E^n$, $g(N_1) \pitchfork N_2$ or $g(N_1)$ is clean to N_2. This implies density on V_i.

Furthermore, for arbitrarily small $g \in O_n \times E^n$, there exists an arbitrarily

small diffeomorphism $\tilde{g}$ on V_i such that $g(N_1) = \sigma_{\tilde{g}(\mathcal{F})}(V_i)$. Such $\tilde{g}$ are isotopic to the identity. (This implies (iii) on V_i.) Let $h_t^i: V_i \hookrightarrow$ be this isotopy. Now $\tilde{g}|_{V_i}$ and h_t^i are extended via a bump function to the rest of M, both equaling the identity outside of U_i. (On $U_i \setminus V_i$ they are bump-averaged with the identity.) This is possible because each is very small.

Next on V_{i+1} an even smaller isotopy, $\tilde{h}_t^{i+1}$, is used which does not disturb the transversality on V_i and which satisfies the same transversality requirements on V_{i+1}. Again this $\tilde{h}_t^{i+1}$ is extended to all of M. Now let $h_t^{i+1} = \tilde{h}_t^{i+1} \circ h_t^i$. In this manner an isotopy is constructed which satisfies (iii). This arbitrarily small isotopy pushes $\mathcal{F}$ into $\mathcal{R}$.

Showing (i)(b).

$$\mathrm{Con}(\mathcal{F}, \mathcal{F}_0) = \left\{ x \in M: \mathcal{C}_{\sigma_0}(x) \cap \sigma_{\mathcal{F}}(x) \neq \varnothing \right\} = \sigma_{\mathcal{F}}^{-1}\left\{ \mathcal{C}_{\sigma_0} \cap \sigma_{\mathcal{F}}(M) \right\}.$$

Since $\sigma_{\mathcal{F}}$ is diffeomorphic onto its image, $\dim \mathrm{Con}(\mathcal{F}, \mathcal{F}_0) = \dim \mathcal{C}_{\sigma_0} \cap \sigma_{\mathcal{F}}(M)$. For $\mathcal{F} \in \mathcal{R}$,

$$\dim \mathcal{C}_{\sigma_0} + \dim \sigma_{\mathcal{F}}(M) - \dim GB_{n,n-k}$$
$$= \left[(k-1)(n-k+1) + n \right] + \left[n \right] - \left[(n-k)k + n \right]$$
$$= 2k - 1.$$

Showing (ii)(b). For $\mathcal{F} \in \mathcal{R}$,

$$\dim \mathrm{Con}(\mathcal{F}, \mathcal{F}_0) = \dim \mathcal{B}_{\sigma_0} + \dim \sigma_{\mathcal{F}}(M) - \dim GB_{n,n-k}$$
$$= \left[(n-k-1)(k+1) + n \right] + \left[n \right] - \left[(n-k)k + n \right]$$
$$= 2(n-k) - 1.$$

Showing (iv) (*All foliations are codimension* 1). Let $\mathcal{G}_0 \in \mathcal{R}(\mathcal{F}_0)$. We show there exists a C^∞-small isotopy which carries $\mathcal{G}_0$ through $\mathcal{R}(\mathcal{F}_0)$ to an element $\mathcal{G}$ of $\mathcal{R}_0$. First, $\mathrm{Con}(\mathcal{F}_0, \mathcal{G}_0)$, being compact, may be covered by finitely many nested triplets of $\mathcal{F}_0$-boxes, $U_i \subset V_i \subset W_i$, $i = 0, \ldots, N$. We assume $\mathrm{Con}(\mathcal{F}_0, \mathcal{G}_0) \subset \bigcup_{i=0}^{N} \mathrm{Int}\, U_i$ and that the ratios of radii of these boxes is fixed for all i, $\frac{3}{4} : \frac{7}{8} : 1$. (Figure 6.) These triplets are chosen so that (a) any $\mathcal{G}_0$-leaf which passes through a point V_i is (in W_i) the graph of a map $g_0: B^{n-1} \to (-1, 1)$ where $\|Dg_0\| \ll 1$, and (b) the union of the $\mathcal{G}_0$-leaves (in W_i) which pass through $V_i \cap (0 \times [-1, 1])$ contains a neighborhood of U_i. Such triplets do exist since on Con, $\mathcal{F}_0$ and $\mathcal{G}_0$ are tangent so that by choosing these triplets very small, g_0 will have as small a derivative as need be. That the ratios of these boxes is kept constant assures that a finite number of such triplets will cover $\mathrm{Con}(\mathcal{F}_0, \mathcal{G}_0)$ in the sense that a finite number of the U_i are a covering.
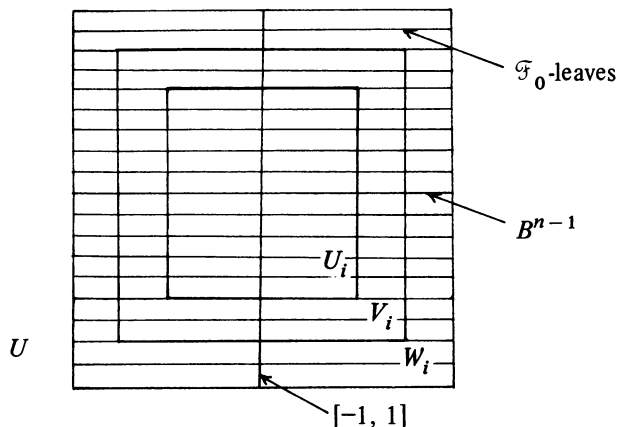
FIGURE 6

Consider $U_0 \subset V_0 \subset W_0$. Let $A = V_0 \cap (0 \times [-1, 1]) = 0 \times [-\frac{7}{8}, \frac{7}{8}]$. g_0 $= g_0(x, \alpha)$, $x \in B^{n-1}$ and $\alpha \in A$ so that the g_0-leaves are $\mathcal{G}_0$ restricted to W_0. We apply jet-transversality as follows: Let

$$Y = B^{n-1} \times A \times L(\mathbf{R}^{n-1}, \mathbf{R}) \times L_s^2(\mathbf{R}^{n-1}, \mathbf{R}).$$

This last factor is the space of bilinear maps: $\mathbf{R}^{n-1} \times \mathbf{R}^{n-1} \to \mathbf{R}$. Let $X = \{(x, \alpha, T, S) \in Y: T = 0 \text{ and } \det S = 0\}$. Then X is a submanifold complex of Y with codimension n; the codimension of $\{T = 0\}$ in $L(\mathbf{R}^{n-1}, \mathbf{R}) \approx \mathbf{R}^{n-1}$ is $n - 1$ and the codimension of $\{\det S = 0\}$ is 1 in $L_s^2(\mathbf{R}^{n-1}, \mathbf{R})$.

Jet-transversality implies now that we can C^∞-approximate g_0 by g so that the jet map $Jg: B^{n-1} \times A \to Y \supset X$ given by

$$Jg: (x, \alpha) \mapsto \left(x, \alpha, d_x g_0(\cdot \, ; \alpha), d_x^2 q_0(\cdot \, ; \alpha)\right)$$

is transverse to X. In particular $(Jg)^{-1}(X)$ has codimension n; hence it is finite. Besides, by transversality $(Jg)^{-1}(X)$ does not change topological type under subsequent small perturbations of g. Furthermore, the points of $(Jg)^{-1}(X) \cap U_i$ are precisely the set of non-Morse contact points in U_i.

Now we observe there exists a C^∞-small isotopy with support in W_0 which vertically pushes the $\mathcal{G}_0|_{U_0}$-leaves to the $g|_{U_0}$-leaves. Thus on U_0, we have our result for a new global foliation $\mathcal{G}_1$, isotopic to $\mathcal{G}_0$ by a small isotopy. Properties of our triplets, (a) and (b), being open, continue to hold for our new $\mathcal{G}_1$. Openness or stability of jet-transversality assures that we may now go on to the next triplet of boxes, perturbing $\mathcal{G}_0$ on them without disturbing our result on U_0. Continuing, the desired $\mathcal{G} \in \mathcal{R}_0$ is attained.

COROLLARY. *Let* $T = T_{\mathcal{G}_0} = \{\mathcal{F} \in \mathrm{Fol}_k^\infty(M): \sigma_{\mathcal{F}} \pitchfork \sigma_{\mathcal{G}_0}\}$. *Then*

(a) *T is open and dense in* $\mathrm{Fol}(M)$, *and*

(b) *For all* $\mathcal{F} \in T_{\mathcal{G}_0}$, $\mathrm{Tan}(\mathcal{F}, \mathcal{G}_0)$ *is the finite union of smooth* $[n - k(n - k)]$-*submanifolds.* (*When* $[n - k(n - k)] < 0$ Tan *is empty.*)

PROOF. For $\mathfrak{F} \in T_{\mathfrak{F}_0}$,

$$\dim(\sigma_0(M) \cap \sigma(M)) = 2n - [(n-k)(n) + n] = n - k(n-k).$$

The rest of the proof follows the above proof of Theorem 1.

REMARKS. (i) Suppose it is known that $\mathrm{Con}(\mathfrak{F}, \mathcal{G})$ has dimension less than $2k - 1$ ($k \leqslant n/2$). One may not conclude that there exists a small isotopy such that $\mathrm{Con}(\mathfrak{F}, h_1(\mathcal{G})) = \varnothing$. (Two spheres in 3-space which intersect in a point may always be perturbed apart.) The following example is due to Hirsch [4]: Let X be a nonvanishing vector field on S^5. $(\mathrm{Im}\, X)^{\perp} \subset TS^5$ is a 9-manifold. Bump X to Y so that Y has one singularity. The projections of Y and the zero section into $(\mathrm{Im}\, X)^{\perp}$ are two 5-submanifolds which intersect in a point. S^5 has at most one linearly independent nonvanishing vector field [12]–so the two 5-submanifolds cannot be perturbed apart.

(ii) In the Franks and Robinson isotopy [1], $n = 3$, $k = 2$. Then generically $\dim \mathrm{Con} = 1$. This is in fact the dimension of Con before their large isotopy.

(iii) Not all the intermediate lower dimensions of submanifolds of the complex Con necessarily occur. These dimensions may be calculated starting with $\dim A_x^s$.

3. Domed contact.

DEFINITIONS. (1) $\mathfrak{F}$ is a *foliation with centers* of M^n if about each point of M there exist an open neighborhood U and Morse function $f\colon U \to \mathbf{R}$ such that the leaves of $\mathfrak{F} \cap U$ are the level surfaces of f and all singularities of f are either index 0 or index n.

(2) Two foliations $\mathfrak{F}$ and $\mathcal{G}$ have *domed contact* if $\mathrm{Con}(\mathfrak{F}, \mathcal{G}) \neq \varnothing$ and all contact is Morse of index 0.

(3) A foliation with boundary $\tilde{\mathfrak{F}}$ of $M \times I$ *bridges* two foliations with centers $\mathfrak{F}$ and $\mathcal{G}$ if (i) $\tilde{\mathfrak{F}} \cap \{M \times \{0\}\} = \mathfrak{F}$ and $\tilde{\mathfrak{F}} \cap \{M \times \{1\}\} = \mathcal{G}$, and (ii) $\tilde{\mathfrak{F}}$ has domed contact with the trivial foliation $\{M \times \{t\}\}$.

(4) Denote $M_t = M \times \{t\}$.

EXAMPLE 1. $\mathcal{G}$ is the trivial foliation of $S^2 \times I$ (consisting of concentric spheres). $\tilde{\mathfrak{F}}$ is a foliation with boundary containing two Reeb sectors. The rest of $\tilde{\mathfrak{F}}$ is filled out by annular leaves. $\tilde{\mathfrak{F}}$ has domed contact with $\mathcal{G}$ along two radial line segments. Everywhere else the two foliations are transverse. $\tilde{\mathfrak{F}} \cap M_0$ and $\tilde{\mathfrak{F}} \cap M_1$ are foliations with centers bridged by $\tilde{\mathfrak{F}}$. Notice that if one of the Reeb sectors were reversed in orientation, this new foliation would still bridge $\tilde{\mathfrak{F}} \cap M_0$ and $\tilde{\mathfrak{F}} \cap M_1$; the isotopy class of the bridging foliation is not unique. (Figure 7.)

THEOREM 2 (DOMED CONTACT). *Let M^n be a compact orientable manifold with boundary, $n > 3$. A proper codimension 1 foliation $\mathfrak{F}$ of M and a codimension 1 foliation with boundary, $\mathcal{G}$ (such that $\mathrm{Con}(\mathfrak{F}, \mathcal{G})$ meets ∂M) have domed*

contact if and only if
  (i) *M is homeomorphic to $S^{n-1} \times I$,*
  (ii) $\mathcal{F}$ *is isotopic to the trivial foliation, and*
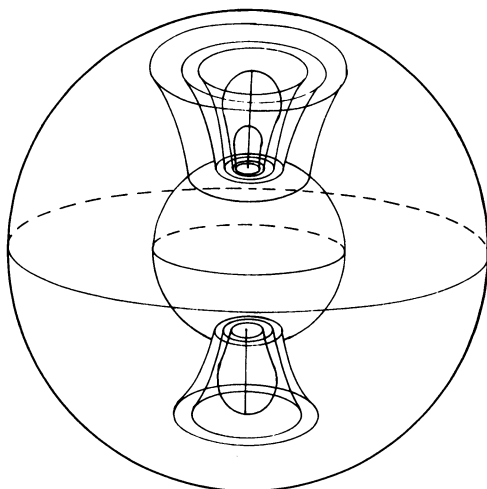 (iii) $\mathcal{G}$ *is in one of five certain isotopy classes.*

FIGURE 7

PROOF. ($\Rightarrow$) (Proof of (iii) due to Glenn Davis.)

Showing (i). Assume $\mathcal{G}$ has domed contact with $\mathcal{F}$ at p_0, an element of the boundary component $\partial_0 M$. Then on a neighborhood of p_0, $U \subset \partial_0 M$, live coordinates $(x_1, \ldots, x_{n-1})$ and a defining function $f: U \rightarrow \mathbf{R}$ of $\mathcal{G} \cap U$ given by $f(x_1, \ldots, x_{n-1}) = x_1^2 + \cdots + x_{n-1}^2$. Leaves near p_0 are $(n-2)$-spheres: $\mathcal{G} \cap U$ is a foliation with one center.

M compact implies that such points of domed contact with $\partial_0 M$ are finite; call them $\{p_i\}_{i=0}$. Near each p_i, extract a small open $(n-1)$-ball, $B_i \subset \partial_0 M$, in such a way that $\mathcal{G}' = \{(\mathcal{G} \cap \partial M) \setminus \cup B_i\}$ is a proper foliation of $V = \partial M \setminus \cup B_i$.

We now show that $V \approx S^{n-2} \times I$. If $n = 3$, V is a 2-manifold with boundary the union of circles. The only such (orientable) foliatable manifold is the cylinder $S^1 \times I$. For $n > 3$, V is an $(n-1)$-manifold with boundary the union of simply connected $(n-2)$-spheres. Let $\tilde{V}$ be the double of V and $\tilde{\mathcal{G}}'$ the double of $\mathcal{G}'$. As a consequence of the Reeb stability theorem [10], [6], since $\tilde{\mathcal{G}}'$ contains a compact leaf with finite fundamental group ($\approx S^{n-1}$), $\tilde{V}$ is a fiber bundle over S^1 with fibers S^{n-2}. Thus the reduced Mayer-Vietoris sequence

$$\cdots \tilde{H}_1(\partial V) \rightarrow \tilde{H}_1(V) \oplus \tilde{H}_1(V) \rightarrow \tilde{H}_1(\tilde{V}) \rightarrow \tilde{H}_0(\partial V) \rightarrow 0$$

has $\tilde{H}_1(\partial V) \approx 0$ and $\tilde{H}_1(\tilde{V}) \approx \mathbf{Z}$. Therefore $\tilde{H}_1(V) \oplus \tilde{H}_1(V) \approx 0$ since no

other subgroup of $\mathbf{Z}$ besides 0 is the direct sum of two identical subgroups of $\mathbf{Z}$. Consequently, $\tilde{H}_0(\partial V) \approx \mathbf{Z}$; ∂V has two components. A topological argument concludes that there exist θ_1 and θ_2 in S^1 such that ∂V is the union of fibers over θ_1 and θ_2. Thus there exist two copies of V in $\tilde{V}$, fiber bundles over the intervals $[\theta_1, \theta_2]$ and $[\theta_2, \theta_1]$ respectively in S^1. $[\theta_1, \theta_2]$ contractible implies that $V \approx S^{n-2} \times I$.

By gluing back the $(n-1)$-balls, we conclude that $\partial_0 M$ is homeomorphic to S^{n-1}, for all $n \geqslant 3$. We repeat the argument to conclude that for $n > 3$, M^n is homeomorphic to $S^{n-1} \times I$. (For $n > 8$, these $(n-1)$-spheres may have exotic differentiable structures, depending on the choice of the gluing map.)

Showing (ii). $\mathcal{F}$ contains the simply connected leaf $M_0 \approx S^{n-1}$. The Reeb stability theorem implies then that there exists a foliation neighborhood U about M_0 of the form $M_0 \times I$ and foliated by $\{M_0 \times \{t\}: t \in I\}$ (for some interval I).

Let $g: M \to [a, b]$ be a Morse function with properties: (1) $g(M_0) = a$, $g(M_1) = b$, and (2) for all leaves $\mathcal{L} \in U$, $g(\mathcal{L})$ is a regular value with level surface $\mathcal{L}$. Now assume there exists $c \in (a, b)$ such that (1) $[a, c)$ are regular values of g, (2) for all $x \in [a, c)]$, $g^{-1}(x)$ is a spherical leaf of $\mathcal{F}$, and (3) c is the least upper bound with respect to (i) and (ii). By continuity, $g^{-1}(c)$ is a leaf of $\mathcal{F}$. So c cannot be a critical value of g. Therefore $g^{-1}[a, c) \approx [a, c) \times S^{n-1}$ and has as one of its boundaries $g^{-1}(c) \approx S^{n-1}$. But the Reeb stability theorem implies there must exist a foliation neighborhood of spherical leaves about $g^{-1}(c)$. So no such $c \in (a, b)$ exists. All the leaves of $\mathcal{F}$ are $(n-1)$-spheres; $\mathcal{F}$ is isotopic to the trivial foliation.

Showing (iii). A similar argument concludes that $\mathcal{G}_0 = \mathcal{G} \cap M_0$ and $\mathcal{G}_1 = \mathcal{G} \cap M_1$ are both isotopic to the standard north pole-south pole foliation (with two centers) of S^{n-1}. We assume then that $\mathcal{G}$ bridges $\mathcal{G}_0$ to itself. By a preliminary isotopy we may assume that $\mathrm{Con}(\mathcal{F}, \mathcal{G}) = (p \times I) \cup (q \times I)$ where p and q are the centers of $\mathcal{G}_1$. Also assume that there exist small disks D_p and D_q about p and q such that $\partial(D_p \times I) \cap \mathcal{G}$ and $\partial(D_q \times I) \cap \mathcal{G}$ are the trivial foliations of $S^{n-2} \times I$ by spheres. Each leaf of this foliation is contained in the same t-level of $M \times I$.

There are three possibilities depending on how the leaves of $\mathcal{G}$ pass through p and q (Figure 8). Recall that the Morse index is constant along $p \times I$ and $q \times I$.

Case 1. We show in this case that $\mathcal{G}$ is isotopic to a particular foliation with boundary, $\mathcal{H}_0$. When $n = 3$, $\mathcal{H}_0$ is the spin of the foliation of the annulus shown in Figure 9. By the Morse contact lemma (and possibly by shrinking D_p and D_q), we may assume that $\mathcal{G}$ and $\mathcal{H}_0$ agree on $(D_p \times I) \cup (D_q \times I)$.

Let $K = (M \times I) \setminus \mathrm{int}(D_p \times I) \setminus \mathrm{int}(D_q \times I)$. So $K \subset M \times I$ and inherits
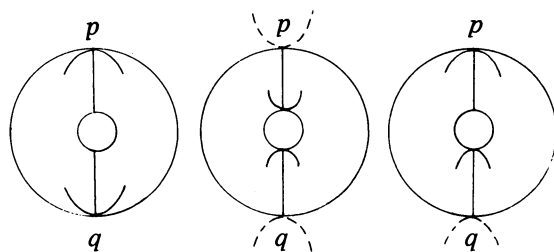
CASE 1   CASE 2   CASE 3

FIGURE 8

coordinates (x, τ). There being no contact, $\mathscr{G}$ and $\mathscr{H}_0$ are both transverse to $\mathscr{F}$ on K. Now construct a vector field $X_{\mathscr{G}}$ on K which is tangent to $\mathscr{G}$ and which projects to $\partial/\partial\tau$ on I. Similarly for $X_0 = X_{\mathscr{H}_0}$. Since $\mathscr{G}$ and $\mathscr{H}_0$ are Case 1, these vector fields point out of K on $\partial(D_p \times I) \cup \partial(D_q \times I)$ (Figure 10). Denote by $\phi_{\mathscr{G}}$ the flow generated by $X_{\mathscr{G}}$, and by ϕ_0 the flow generated by X_0.
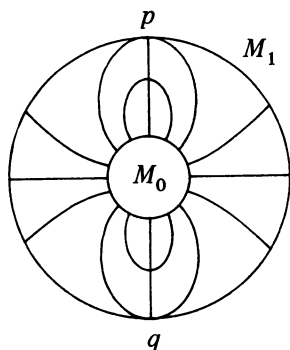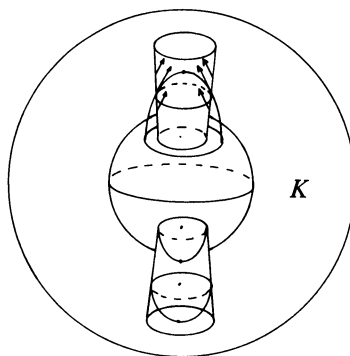

FIGURE 9

FIGURE 10

Let $N = [M_1 \setminus (D_p \times \{1\}) \setminus (D_q \times \{1\})] \cup (\partial D_p \times I) \cup (\partial D_q \times I)$. It is important to note that $\mathscr{G}$ and $\mathscr{H}_0$ agree on N. Define $g_t : K \hookleftarrow$, $t \in [0, 1]$, as follows: If $\tau > t$, $g_t(x, \tau) = (x, \tau)$. If $\tau \leqslant t$, flow along $\phi_{\mathscr{G}}$ from (x, τ) until either the I-coordinate has reached t or N is encountered. Then flow in negative time by ϕ_0 until the I-coordinate is once again τ. This point is defined to be $g_t(x, \tau)$. The reverse operation describes g_t^{-1}; $\mathscr{G}$ being of Case 1 assures that this operation is well defined. It follows that g_t is an isotopy with $g_0 = \mathrm{id}_K$.

g_t fixes N for all t. Since $\partial(D_p \times I) \cup \partial(D_p \times I) \subseteq N$, the map $G = g_1 \cup \mathrm{id}_{D_p \times I \cup D_q \times I}$ is well defined. Then G is isotopic to the identity (through g_t). Since $\mathscr{G}$ and $\mathscr{H}_0$ agree on N, g_1 and hence G takes leaves of $\mathscr{G}$ to leaves of $\mathscr{H}_0$. (Notice that g_t $(t < 1)$ does not necessarily take leaves to leaves since

some pairs of points are sent up to the t level but then are reversed down different leaves of $\mathcal{H}_0$.) Thus $\mathcal{G}$ and $\mathcal{H}_0$ are isotopic bridging foliations.

Case 2. A reflection of Case 1.

Case 3. In this case, we show $\mathcal{G}$ is contained in one of three isotopy classes. Again define $\phi_\mathcal{G}$ on K (cf. Figure 11.) Let

$$S_p = \{\text{leaf of } \mathcal{G} \text{ passing through } p \times 1\} \cap (\partial D_p \times I),$$

$$S_q = \{\text{leaf of } \mathcal{G} \text{ passing through } q \times 0\} \cap (\partial D_q \times I).$$

So S_p and S_q are $(n - 2)$-spheres at constant τ-levels. Let $\tau_p = \{I\text{-coordinate}$ of $S_p\}$. Consider the evolution of S_q under $\phi_\mathcal{G}$. Points flow out of $D_q \times I$ under $\phi_\mathcal{G}$, so S_q cannot return to $\partial(D_q \times I)$. Since $\phi_\mathcal{G}$ has unit speed in the $\partial/\partial\tau$- or I-direction, S_q maintains constant τ-coordinate at each forward time. Thus there are three possibilities: S_q flows into N (i) with I-coordinate $> \tau_p$, (ii) with I-coordinate $< \tau_p$, or (iii) with I-coordinate $= \tau_p$.
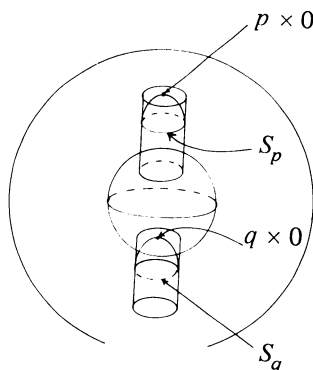

FIGURE 11

Consider case (i). If this I-coordinate is < 1, then S_q must have hit $\partial(D_p \times I)$ "above" S_p, say at S_0. S_0 is contained in a leaf of $\mathcal{G}$ which intersects M_1 near p. (Figure 12.) So if D_p (and hence S_p) is chosen smaller this forward saturation of S_q would not intersect $\partial(D_p \times I)$. We assume that S_q hits $N \cap M_1$ in forward time. The picture is exaggerated as in Figure 13.

We now show that $\mathcal{G}$ is isotopic to a foliation with boundary, $\mathcal{H}_1$. When $n = 3$, $\mathcal{H}_1$ is the spin of the following foliation of the annulus. (Figure 14.)

Let A be the saturation of $\partial(D_q \times I)$ under $\phi_\mathcal{G}$. Let U be an ε-open neighborhood of A. Define $g_t': U \to K$ as follows: If $\tau \leqslant 1 - t$, $g_t'(x, \tau) = (x, \tau)$. If $\tau \geqslant 1 - t$ flow backwards along $\phi_\mathcal{G}$ until either the I-coordinate reaches $1 - t$ or until N is encountered. Then flow forward along $\phi_{\mathcal{H}_1}$ until the I-coordinate is again τ. This point is $g_t'(x, \tau)$. Again g_t' is an isotopy of U with $g_1' = \text{id}_U$ which carries leaves of $\mathcal{G}$ to leaves of $\mathcal{H}_1$.

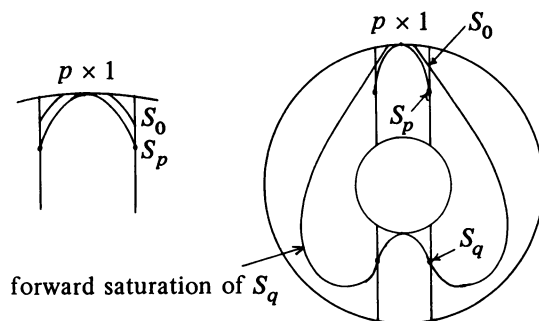Now apply the isotopy extension theorem (Hirsch, [3, p. 180]) to get an
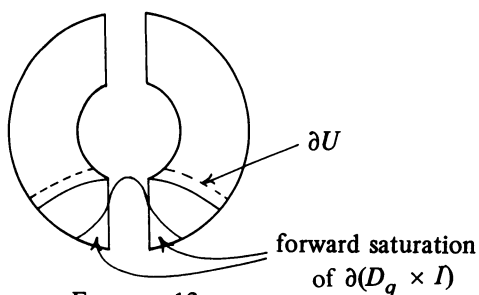
cross-section of $M \times I$

FIGURE 12

FIGURE 13

isotopy of K which carries leaves of $\mathcal{G}$ contained in A to leaves of $\mathcal{K}_1$. We now make the assumption that $\mathcal{G}$ and $\mathcal{K}_1$ are identical on A as well as $\phi_{\mathcal{G}}$ and $\phi_{\mathcal{K}_1}$. Hence the isotopy $g_t: K \hookleftarrow$ defined as in Case 1 is the identity on A. Once again g_1 is extended to G; $\mathcal{G}$ and $\mathcal{K}_1$ are isotopic bridging foliations.

Case (ii) (S_q *flows into N with I-coordinate* $< \tau_p$). In this case a similar proof shows that $\mathcal{G}$ is isotopic to a foliation $\mathcal{K}_2$. For $n = 3$, this is the spin of the foliation of the annulus shown in Figure 15.
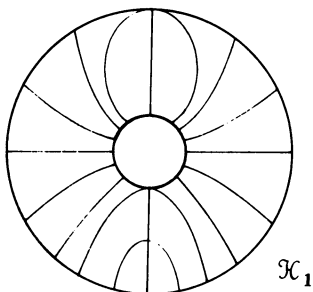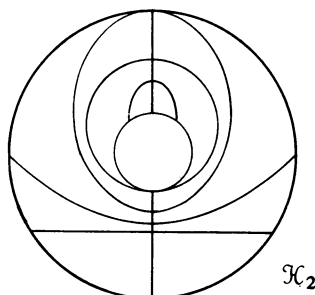

FIGURE 14

FIGURE 15

Case (iii) (*I-coordinate* $= \tau_p$). In this case $\mathcal{G}$ is isotopic to a "degenerate" foliation $\mathcal{K}_3$; see Figure 16.
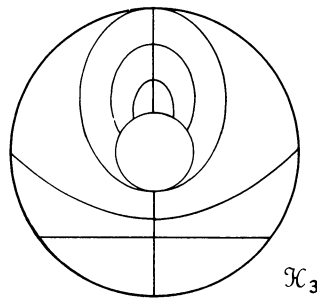
FIGURE 16

($\Leftarrow$) Knowing that $\mathcal{G}$ is isotopic to $\mathcal{H}_0$, a reflection of $\mathcal{H}_0$, $\mathcal{H}_1$, or $\mathcal{H}_3$ makes this implication trivial. $\square$

REMARK. If $n = 3$ then (i) and (ii) follow identically. But the isotopy classes of $\mathcal{G}$ are infinite in number. This is because the Reeb stability theorem does not apply to nonsimply connected leaves, in this case, to circle leaves. These circle leaves may bound Reeb components.

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